Rayleigh Instability of an Annulus

Studying capillary break-up of an annulus utilizing the surface energy argument and the linear stability analysis technique

Amy Nelson

for

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Dr. Roger Bonnecaze

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Introduction

A slow-moving, thin cylindrical stream of water will become non-uniform in diameter and eventually break-up into droplets. This is an example of capillary break-up and commonly known as Rayleigh instability. For a cylindrical jet, Lord Rayleigh calculated that the most unstable disturbance wavelength is about nine times the radius of the jet. This "most dangerous mode" ($\lambda = 9.016*r$) has been proven experimentally and can be derived via linear stability analysis.

The purpose of this project was to predict the most unstable disturbance wavelength for liquid jet of varying annular geometry ($0 < \kappa < 1$).

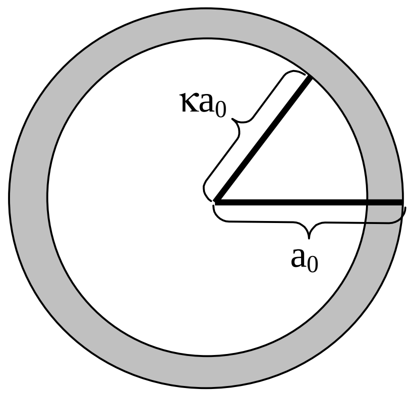


Figure 1. Annulus with outer diameter a_0

Surface Energy Argument

A disturbance will grow only if the process is energetically favorable. The surface energy argument says that if a disturbance increases the surface energy of the system, that disturbance will decay due to restoring forces (viscosity), which will bring the system to its lowest energy state. Figure 1 shows the cross section of the unperturbed annulus with outer radius, a_0. Figure 2 shows the lengthwise view of the jet of constant thickness with a sinusoidal disturbance.

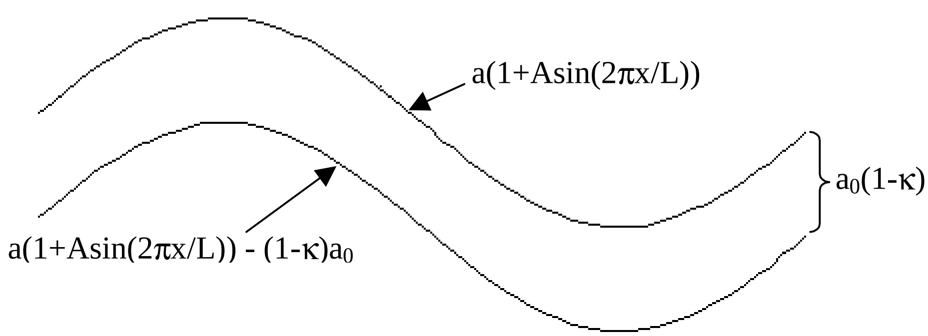


Figure 2. Lengthwise view of disturbed system

For an inviscid, one-dimensional annular jet, the surface energy argument utilizes the following equation to establish a relationship between a and a_0 (see figure 2 for definition of a and a_0). V is the volume of the fluid, x is the axial distance, L is the wavelength of the disturbance, *A* is the amplitude of the disturbance, and κ is the ratio of the inner radius to the outer radius.

$$V = \int_0^L \pi \left(1-\kappa^2\right) a^2 \left(1 + A\sin\left(\frac{2\pi x}{L}\right)^2\right) dx = a_0\left(1-\kappa^2\right)\pi L$$
$$a = a_0\left(1 + \frac{1}{2}A^2\right)^{-1/2} \tag{1}$$

The expression for the surface area of the fluid is as follows.

$$S = 2\times\int_0^L 2\pi\, a\left(1 + A\sin\left(\frac{2\pi\, x}{L}\right)\right)\left(1 + \left(\frac{2\pi a A}{L}\right)^2 \cos^2\left(\frac{2\pi x}{L}\right)\right)^{1/2} dx$$
$$-\int_0^L 2\pi(1-\kappa)\, a_0\, dl \tag{2}$$

Now, to calculate the change in surface energy the following equation applies where *E* is the surface energy of the system, γ is the surface tension of the fluid, and S_0 is the surface area of the unperturbed system.

$$\Delta E = \gamma\left(S - S_0\right)$$
$$S_0 = 2\pi L a_0(1-\kappa) \tag{3}$$

For details of the calculations, see the attached appendix.

Linear Stability Analysis

A linear stability analysis provides more information about the disturbance. Information such as whether or not a disturbance will grow and the value of the most unstable wavelength can be obtained. If only small perturbations are of interest, terms involving the squares of or products of the disturbance amplitude can be neglected therefore making the equations linear. The analysis was based on the following assumptions:

a. The fluid is inviscid and incompressible
b. The disturbance amplitude is very small (Amplitude, A <<1).
c. Frame of reference effects are neglected.

The analysis utilized the following equations.

Conservation of Mass:

$$\underline{\nabla} \bullet \underline{u}' = 0 \tag{4}$$

Here $\underline{u}'$ is the velocity of the fluid neglecting frame of reference effects.

Conservation of Momentum:

$$\rho \frac{\partial \underline{u}'}{\partial t} = -\underline{\nabla}\, p' \tag{5}$$

Here ρ is the density of the fluid and p' is the pressure in the fluid.

Kinematic Expression:

$$u_r' = \frac{\partial \eta}{\partial t} \tag{6}$$

Here η is the length of the disturbance and u_r' is the radial velocity of the fluid.

For a sinusoidal disturbance p', η, $\underline{u}'$ and can all be described by B in the following expression:

$$B = \hat{B}\ \exp(st + i(k\,z + n\,\theta))$$

Here, s is the time factor, k is the wave number ($k = 2\pi/\lambda$ where λ is the wavelength of the disturbance), and n is the angular number that determines the shape of the disturbance. The value of n is assumed to be zero (symmetrical disturbance) for this study to eliminate any imaginary component of the solution. Combining equations 4 and 5 yields the following differential equation whose solution is a zero order modified Bessel's equation. The order of the Bessel equation is determined by the value of n.

$$r^2 \frac{d^2\hat{p}}{dr^2} + r\frac{d\hat{p}}{dr} - (r^2 k^2 + n^2)\,\hat{p} = 0 = \underline{\nabla}^2 p' \tag{6}$$

$$\hat{p}\,(r) = AI_n(kr) + BK_n(kr) \tag{7}$$

Boundary Conditions

The following boundary conditions were used to solve the differential equation. This first boundary condition requires that the normal stresses be balanced at the outer interface. This equation is also known as the Young-Laplace equation where $\underline{n}$ is the normal vector, γ is the surface tension of the fluid, and a is the unperturbed outer radius of the annulus (equivalent to a_0 in figure 1).

$$p'\Big|_{r=a+\eta} = -\gamma\left(\frac{\eta}{a^2} + \frac{\partial^2\eta}{\partial z^2} + \frac{1}{a^2}\frac{\partial^2\eta}{\partial\theta^2}\right) = \underline{\nabla} \bullet \underline{n} \tag{8}$$

The second boundary condition comes from the fact that the radial velocity at the outer interface is equal to the time derivative of the length of the disturbance (equation 6).

$$\hat{u}_r\Big|_{r=a+\eta} = \frac{Ak}{\rho s}\ I_0'(ka) + \frac{Bk}{\rho s}\ K_0'(ka) = \frac{\partial\eta}{\partial t} \tag{9}$$

A third boundary condition is needed to achieve three simultaneous equations from which to solve for *A*, *B*, and $\hat{\eta}$. If it is assumed that the perturbations are coupled and the wall of the annulus is of constant thickness (boundary condition on which the surface energy argument was based), then the third boundary condition requires that the radial velocity of the inner surface equal the radial velocity of the outer surface.

$$\hat{u}_r\Big|_{r=\kappa a+\eta} = \frac{Ak}{\rho s}\, I_0'(\kappa ka) + \frac{Bk}{\rho s}\, K_0'(\kappa ka) = \frac{\partial \eta}{\partial t} \qquad (10)$$

However, if it is assumed that the inner wall of the annulus remains stationary resulting in a non-constant wall thickness, then the final boundary condition becomes much simpler.

$$\hat{u}_r\Big|_{r=\kappa a+\eta} = 0 \qquad (11)$$

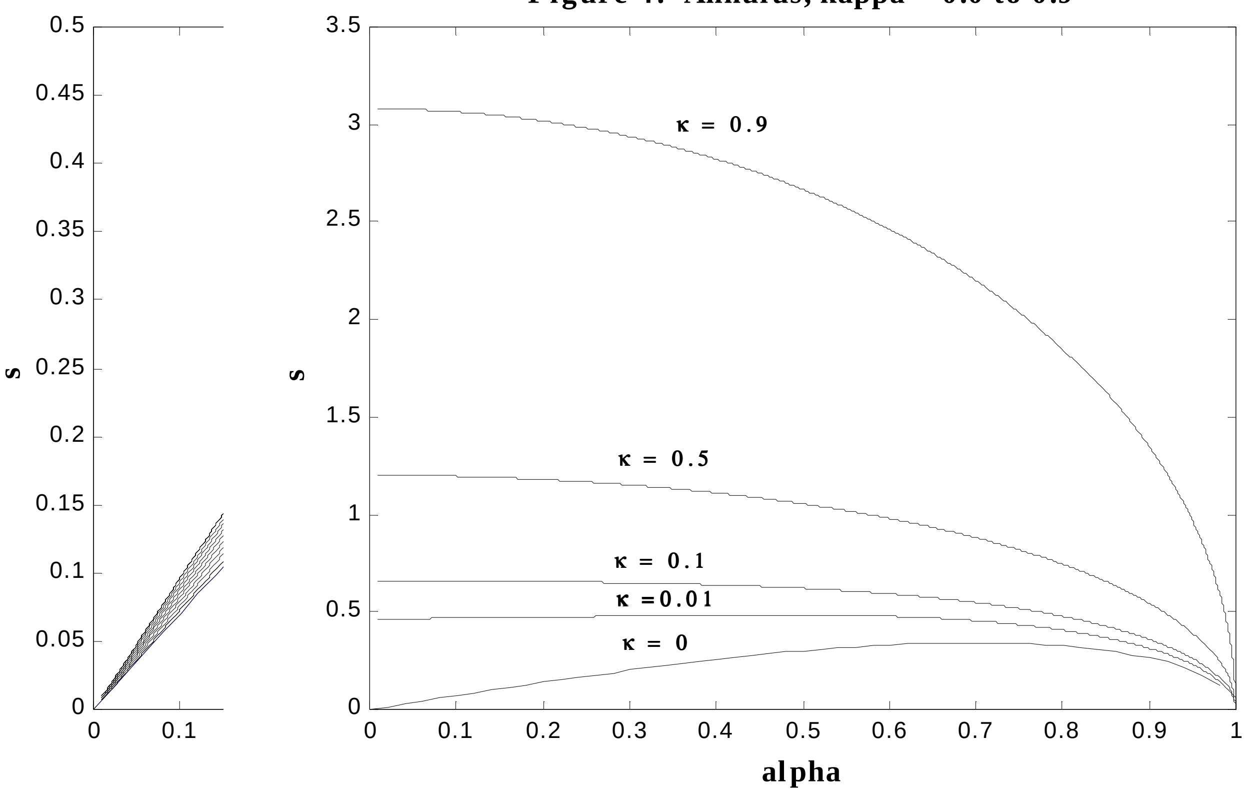
Results

For the base case system (cylindrical, no annulus) the surface energy argument shows that a disturbance will grow for wavelengths greater than $2\pi a_0$ where a_0 is the unperturbed radius of the cylinder. The results of the surface energy argument for the annular case show that this critical wavelength does not change. Therefore, there is no κ dependence on the critical wavelength.

The equations derived from the linear stability analysis were solved for the time factor. The time factor, *s*, was solved with two different sets of boundary conditions. The result for *s* assuming coupled perturbations is as follows. Detailed calculations can be found in the appendix. Here, α is *ka*. A plot of α versus *s* is shown in Figure 3.

$$s = \sqrt{\frac{\dfrac{\gamma\alpha(1-\alpha^2)K_0'(\kappa\alpha)I_0'(\alpha)(1-K_0'(\alpha))}{\rho a^3 I_0(\alpha)I_0'(\kappa\alpha)}}{\left(\dfrac{I_0'(\alpha)}{\ldots}-1\right)\left[\dfrac{K_0(\alpha)}{\ldots}-\dfrac{K_0'(\kappa\alpha)}{\ldots}\right]+\dfrac{K_0'(\kappa\alpha)I_0'(\alpha)-K_0'(\kappa\alpha)}{\ldots}}} \qquad (12)$$

Figure 4: Annulus, kappa = 0.0 to 0.9



Conclusion

It can be seen in both the cases that the critical wavelength ($\lambda_c = 2\pi a$) hasn't changed from the base case ($\kappa = 0$). This is in agreement with the surface energy argument. For a jet of cylindrical geometry ($\kappa = 0$), the most dangerous mode occurs at $s = 0.6203$, $\alpha = 0.697$, $\lambda = 9*a$. The first case where the perturbations are assumed to be coupled, the wavelength at which this maximum occurs doesn't change much. However, it can be seen that the time factor of the disturbance increases as κ is increased (thinner wall). This corresponds to a faster growing disturbance. It can be concluded then, that for a coupled disturbance (which is the most reasonable assumption for an inviscid case), the jet will break up into droplets of equal length as the cylindrical case but it will break up more quickly.

In the case where the inner wall is stationary, the wall is not of uniform thickness. Figure 4 show that the maximum disturbance occurs at infinite wavelength as κ is increased from zero. In this case, the longer wavelength the disturbance, the more quickly the disturbance grows. This case is the less feasible of the two cases because in the case of an inviscid fluid, there will be a disturbance imposed on the inner surface. The former case will most closely approach reality.

References

1. Miller, C., *Interfacial Phenomena*, Marcel Dekker (1985)
2. Abramowitz, M. and Stegun, I., *Handbook of Mathematical Functions*, Dover Publications (1965)